数理解析研究所講究録
第 1555 巻 2007 年 70-72

On quasi-minimal ω-stable groups

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Abstract

Itai and Wakai investigated some group as an example of qusai-minimal structures [1]. We try to characterize such groups more.

1 Quasi-minimal structures and groups

We recall the definition of quasi-minimality. The notion of quasi-minimality is a generalization of that of strong minimality.

Definition 1 An uncountable structure M is called *quasi-minimal* if every definable subset of M with parameters is at most countable or co-countable.

Itai, Tsuboi and Wakai investigated quasi-minimal structures [2]. After that Itai and Wakai showed an example of such structures [1]. They characterized the group (Q^{ω}, $+$, σ, 0) where Q is the set of rational numbers and σ is the shift function.

Definition 2 A function σ is a *shift function* if $\sigma : Q^{\omega} \longrightarrow Q^{\omega}$ and for $\bar{x} = (x_0, x_1, x_2, \cdots\cdots) \in Q^{\omega}$, $\sigma(\bar{x}) = (x_1, x_2, x_3, \cdots\cdots) \in Q^{\omega}$.

They showed that the theory Th(Q^{ω}, $+$, σ, 0) is ω-stable and has the elimination of quantifiers. Thus I tried to characterize structural properties of quasi-minimal ω-stable groups.

2 Quasi-minimal ω-stable groups

(Q^{ω}, $+$) is a divisible abelian group. And it is known that its theory is strongly minimal. So I wondered whether quasi-minimal groups are abelian. By using known Facts about stable groups, it is shown that quasi-minimal nonabelian groups have the strict order property substantially.

Definition 3 A formula $\varphi(x, y)$ has the *strict order property* if there are a_i $(i < \omega)$ such that for any $i, j < \omega$, $\models \exists x\, [\neg\varphi(x, a_i) \wedge \varphi(x, a_j)] \Longleftrightarrow i < j$. A theory T has the *strict order property* if some formula $\varphi(x, y)$ has the strict order property.

Proposition 4 *Let G be a quasi-minimal group. And let Z be the center of G. If G/Z is not abelian, then $Th(G)$ has the strict order property.*

Proof. Suppose that G/Z is nonabelian. As Z is definable subgroup of G, $|\, Z \,|$ is countable. For $a \in G - Z$, let $C_a = \{g \in G \mid a^g = g^{-1}ag = a\}$. Since C_a is definable subgroup of G, $|\, C_a \,|$ is countable. Thus the orbit of a, denoted by $O(a)$, is uncountable set. As orbits are definable equivalence classes, G has only one infinite orbit. In the following, let G be G/Z for convenience of notation. Hence now G has only one nontrivial orbit. So there is $a \in G$ with $a \neq a^{-1}$. As $a^{-1} \in O(a)$, there is $b \in G$ such that $a^b = a^{-1}$. Let $C_G(b) = \{g \in G \mid g^b = g\}$. Since $a^{b^2} = a$ and $a^b \neq a$, $C_G(b^2) \supsetneq C_G(b)$. As $b \in O(a)$, $b^2 \neq 1$ and there is $c \in G$ such that $b^c = b^2$. Then we get $C_G(b) < C_G(b^c) < C_G(b^{c^2}) < \cdots\cdots$. ∎

Thus we can see that quasi-minimal simple (in stability theoretic meaning) groups are abelian essentially.
However, strongly minimal groups and ω-stable abelian groups were characterized completely.

Theorem 5 (Reineke [3]) *Let G be a group. Then the followings are equivalent ;*
(1) G is strongly minimal.
(2) G is minimal.
(3) G is abelian and has the form $G = \oplus_\alpha Q \oplus \oplus_p Z_{p^\infty}^{\beta_p}$ where $\alpha \geqq 0$, β_p is finite, or the form $G = \oplus_\gamma Z_p$ where γ is infinite.

Theorem 6 (Macintyre [4]) *Let G be an abelian group. Then Th(G) is totally transcendental if and only if G is of the form $D \oplus H$ where D is divisible and H is of bounded order.*

And by the following facts about infinite abelian groups, we can see that ω-stable abelian groups are direct sums of strongly minimal groups. (These facts are well known, see e.g. [5]. In them, groups means abelian groups.)

Fact 7 *Let G be a group. Then G has the maximal divisible direct summand.*

Fact 8 *Let G be a divisible group. Then G has the form $G = \oplus_\alpha Q \oplus \oplus_p Z_{p^\infty}^{\beta_p}$.*

Fact 9 *Let G be a group of bounded order. Then G is a direct sum of cyclic groups.*

But we can easily check that ω-stable abelian groups $G = D \oplus H$ in which H has infinitely many summands are not quasi-minimal. Then

Conclusion

Quasi-minimal ω-stable pure groups (i.e. groups reduced to the group language) are strongly minimal substantially.

Thus we should put the next problem last.

Problem

Find quasi-minimal non-ω-stable groups.

References

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